# Gödel and the Limits of Logic

*Mathematical genius Kurt Gödel was devoted to rationality in his work but struggled with it in his personal life*

by John W. Dawson, Jr.

The man in the photograph seen at the right looks formal, reserved and somewhat undernourished. His face and his writings are unfamiliar to most, except for a few philosophers and mathematical logicians. He was Kurt Gödel, celebrated for his incompleteness theorems, the implications of which are far-reaching for the foundations of mathematics and computer science. The story of his life and work is that of a persistent quest for rationality in all things, pursued against a background of recurrent mental instability.

Gödel proved that the mathematical methods in place since the time of Euclid were inadequate for discovering all that is true about the natural numbers. His discovery undercut the foundations on which mathematics had been built up to the 20th century, stimulated thinkers to seek alternatives and generated a lively philosophical debate about the nature of truth. Gödel's innovative techniques, which could readily be applied to algorithms for computations, also laid the foundation for modern computer science.

Born on April 28, 1906, in Brno, Moravia, Gödel was the second of two children of Rudolf and Marianne Gödel, expatriate Germans whose families were associated with the city's textile industry. There were no scholars among Gödel's forebears, and his father received only a trade school education. But being ambitious and hardworking, Rudolf Gödel rose through the ranks to become manager and then part-owner of one of Brno's large textile factories. Along the way he acquired wealth enough to purchase a villa in a fashionable neighborhood and to send his children to private, German-language schools, where both sons did very well in their studies.

Indeed, only once during his primary and secondary school career did young Kurt ever receive less than the highest mark in any subject (mathematics!). Yet he gave no early intimation of genius. He was a highly inquisitive child, so much so that he was nicknamed *der Herr Warum* ("Mr. Why"). But he was also introverted, sensitive and somewhat sickly. At about the age of eight he contracted rheumatic fever, and although it seems not to have caused lasting physical damage, it kept him out of school for some time and may have fostered the exaggerated concern for his health and diet that was to become increasingly prominent over the years.

### The Introvert

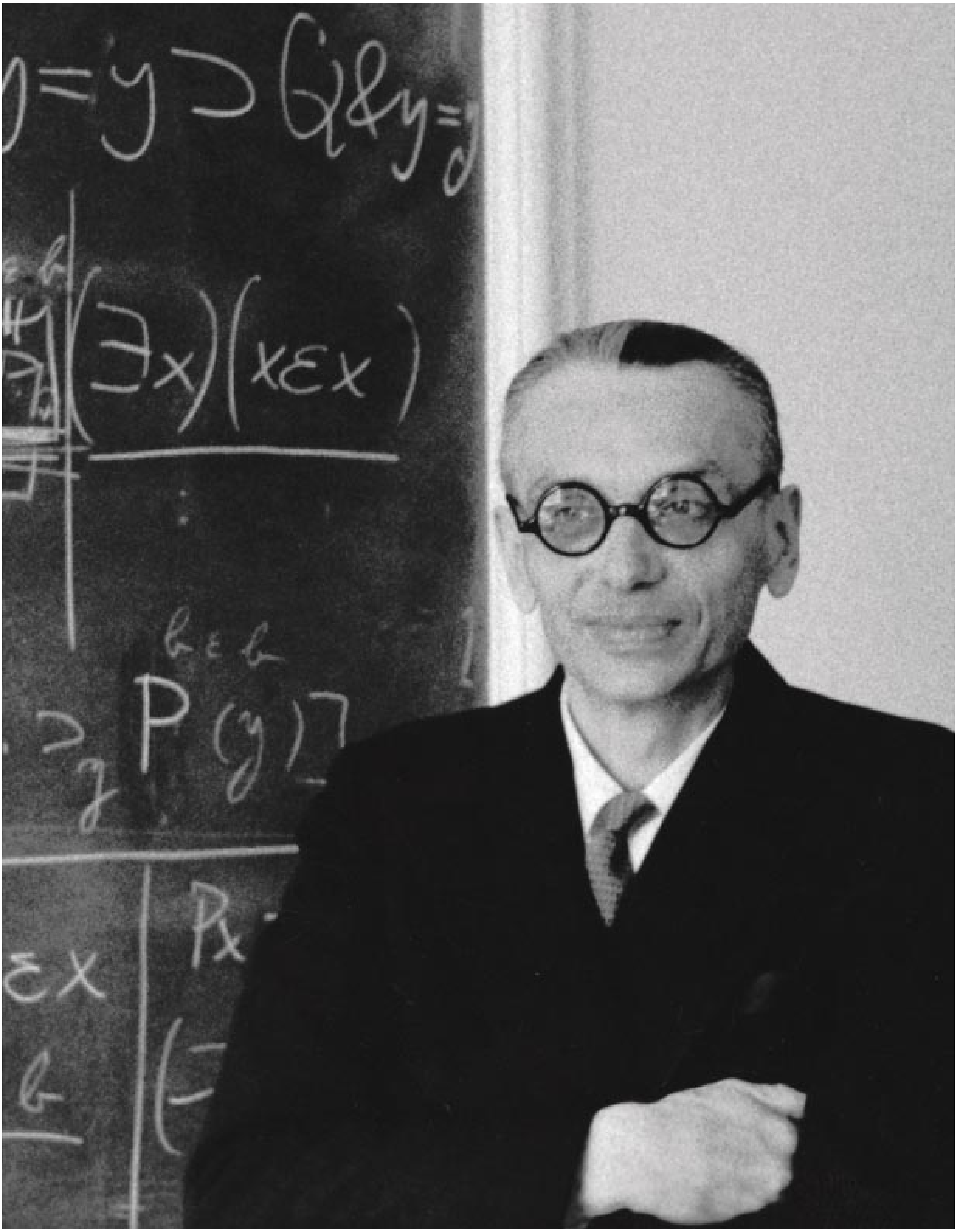
In 1924, after his graduation from the *Realgymnasium,* or technical high school, in Brno, Gödel left his homeland to enroll at the University of Vienna, where his brother had gone four years earlier to pursue medical studies. Vienna's economy was then in ruins. The university, however, retained much of its former eminence. So despite the material privations, during the years between the two world wars Vienna was home to a dramatic flowering of creativity in science, the arts and philosophy.

At the time of his enrollment Gödel intended to seek a degree in physics. But after a short while, impressed by the lectures of professors Philipp Furtwängler and Hans Hahn, he switched to mathematics. His remarkable talents soon attracted attention—so much so that just two years after his matriculation he was invited to attend sessions of a discussion group that Hahn and philosopher Moritz Schlick had founded two years earlier. The group, which was later to become famous as the Vienna Circle, was inspired by the writings of Ernst Mach, a champion of rationalism who believed that all things could be explained by logic and empirical observation, without recourse to metaphysical agencies.

The Circle brought Gödel into contact with scholars such as philosopher of science Rudolf Carnap and mathematician Karl Menger and helped to acquaint him with the literature of mathematical logic and philosophy. In particular, the Circle was immersed in the writings of Ludwig Wittgenstein, whose concern about the extent to which language can speak about language may have prompted Gödel to probe analogous questions about mathematics. Some of the Circle's members, including Carnap, Hahn and physicist Hans Thirring, were active in the investigation of parapsychological phenomena—matters in which Gödel, too, exhibited a keen interest. (Years later he remarked to his close friend, economist Oskar Morgenstern, that in the future it would be deemed a great oddity that 20th-century scientists had discovered the elementary physical particles but had failed even to consider the possibility of elementary psychic factors.)

Gödel did not, however, share the positivistic philosophical outlook of the Circle, which extended Mach's ideas. Instead he was a Platonist: he believed that in addition to objects, there exists a world of concepts to which humans have access by intuition. Thus, for him a statement would have a definite "truth value"—be true or not—whether or not it had been proved or was amenable to being empirically confirmed or refuted. In his own view, that philosophy was an aid to his remarkable mathematical insights.

Although Gödel was an attentive observer and clearly brilliant, he rarely contributed to the Circle's discussions, unless they were about mathematics. Shy and reclusive, he had few close friends. (He did, however, like the company of women and was apparently quite attractive to them.) After 1928 he seldom



COURTESY OF THE ARCHIVES OF THE INSTITUTE FOR ADVANCED STUDY

**KURT GÖDEL proved that mathematical systems are essentially incomplete: not everything that is true can be proved to be so. In later life he turned his attention to a variety of other problems, including relativity. This photograph was taken in May 1958 in Gödel's office at the Institute for Advanced Study in Princeton, N.J., by Finnish logician Veli Valpola.**

COURTESY OF JOHN W. DAWSON, JR.

**TWO BROTHERS, Kurt (*right*) and Rudolf, were close when they were young but as adults drifted apart. This studio shot is from around 1908.**

attended the group's meetings but became active instead in a mathematical colloquium organized by Menger. Its proceedings were published as an annual journal, which Gödel helped to edit and to which he later contributed more than a dozen articles.

### A Reticent Genius

During this period, Gödel suddenly acquired international stature in mathematical logic. Two papers in particular thrust him into prominence. One was his doctoral dissertation, submitted to the University of Vienna in 1929 and published the next year. The other was his treatise "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems," published in German in 1931 and submitted as his *Habilitationsschrift* (qualifying dissertation for entrance into the teaching profession) in 1932.

The dissertation, entitled "The Completeness of the Axioms of the First-Order Functional Calculus," solved an open problem that David Hilbert and Wilhelm Ackermann had posed in their 1928 textbook *Grundzüge der theoretischen Logik* ("Foundations of Theoretical Logic"). The question was whether the accepted rules, stated in the book, for manipulating expressions involving logical connectives ("and," "or" and so on) and quantifiers ("for all" and "there exists," applied to variables that ranged over numbers or sets) would, when adjoined to the axioms of a mathematical theory, enable the deduction of all and only those statements that held true in every structure that satisfied the axioms. In plain words, could one actually prove everything that was true under all interpretations of the symbols?

The expected answer was yes, and Gödel confirmed that it was. His dissertation established that the principles of logic developed up to that time were adequate for their intended purpose, which was to prove everything that was true on the basis of a given set of axioms. It did not show, however, that every true statement concerning the natural numbers could be proved on the basis of the accepted axioms of number theory.

Those axioms, proposed by Italian mathematician Giuseppe Peano in 1889, include the principle of induction. It asserts that any property that is true of zero, and true of a natural number $n + 1$ whenever true of $n$, must be true of all natural numbers. Sometimes called the domino principle—because if you knock one over, the rest will topple—the axiom might seem self-evident. Yet mathematicians found it problematic because it refers not just to numbers themselves but to properties of numbers. Such a "second-order" statement was thought too vague and ill defined to serve as a basis for the theory of natural numbers.

As a result, the induction axiom was recast as an infinite schema of similar axioms that refer to specific formulas rather than to general properties of numbers. Unfortunately, those axioms no longer uniquely characterize the natural numbers, as Norwegian logician Thoralf Skolem demonstrated a few years before Gödel's work: other structures satisfy them as well.

Gödel's completeness theorem states that one can prove all those statements that follow from the axioms. There is a caveat, however: if some statement is true of the natural numbers but is not true of another system of entities that also satisfies the axioms, then it cannot be proved. That did not seem to be a serious problem, because mathematicians hoped that entities that masqueraded as numbers but were essentially different from them did not exist. So Gödel's next theorem came as a shock.

In his 1931 paper Gödel showed that some statement that is true of the natural numbers must fail to be provable. (That is, objects that obey the axioms of number theory but fail to behave like the natural numbers in some other respects do exist.) One could escape this "incompleteness theorem" if all true statements were taken to be axioms. In that case, however, deciding whether some statements are true or not becomes a priori problematic. Gödel showed that whenever the axioms can be characterized by a set of mechanical rules, it does not matter which statements are taken to be axioms: if they are true of the natural numbers, some other true statements about those numbers will remain unprovable.

In particular, if the axioms do not contradict one another, then that fact itself, suitably encoded as a numerical statement, will be "formally undecidable"—neither provable nor refutable—on the basis of those axioms. Any proof of consistency must therefore appeal to stronger principles than the axioms themselves. (For an elucidation of the arguments, see "Gödel's Proof," by Ernest Nagel and James R. Newman; SCIENTIFIC AMERICAN, June 1956.)

The latter result greatly dismayed David Hilbert, who had envisioned a program for securing the foundations of mathematics through a "bootstrapping" process, by which the consistency of complex mathematical theories could be derived from that of simpler, more evident theories. Gödel, on the other hand, saw his incompleteness theorems not as demonstrating the inadequacy of the axiomatic method but as showing that the derivation of theorems cannot be completely mechanized. He believed they justified the role of intuition in mathematical research.

The concepts and methods Gödel introduced in his incompleteness paper are central to the discipline of recursion theory, which underlies all of modern computer science. Extensions of his ideas have allowed the derivation of several other results about the limits of computational procedures. One is the unsolvability of the "halting problem"—that of deciding, for an arbitrary computer with an arbitrary input, whether the computer will eventually halt and produce an output rather than becoming stuck in an infinite loop. Another is the demonstration that no program that does not alter a computer's operating system can detect all programs that do (viruses).

### Shelter in America

Gödel spent the 1933–34 academic year in Princeton, N.J., at the newly founded Institute for Advanced Study, where he lectured on his incompleteness results. He was invited to come there the next year as well but suffered a mental breakdown shortly after his return to Vienna. He recovered in time to return to Princeton in the fall of 1935, but a month after his arrival he experienced a relapse and did not return to lecturing until the spring of 1937 in Vienna.

Without access to Gödel's confidential medical records (he was counseled by a psychiatrist in Princeton), his actual diagnosis must remain unknown. His problems seem to have started with hypochondria: he was obsessive about his diet and bowel habits and kept a daily record for two decades or more of his body temperature and milk of magnesia consumption. He had a fear of accidental and, in later years, deliberate poisoning. This phobia led him to avoid eating food, so that he became malnourished. At the same time, though, he ingested a variety of pills for an imaginary heart problem.

Except in times of crisis, Gödel's mental problems hampered his work surprisingly little. The person who kept him going was Adele Porkert, whom he had met at a Viennese nightclub during his student years. Porkert was a Catholic divorcée six years older than Gödel, who worked as a dancer and whose face was disfigured by a port-wine-stain birthmark. His parents regarded her as a scandalous person. But the two were devoted to each other, and more than once, by serving as his food taster, she helped to allay Gödel's growing fears that someone was trying to poison him. After a long courtship the two were married in September 1938, just before Gödel returned once more to America, where he lectured at the Institute for Advanced Study and at the University of Notre Dame on exciting new results he had obtained in set theory.

That achievement involved the resolution of some controversial aspects of the theory of collections of objects. In the late 19th century, German mathematician Georg Cantor had introduced a notion of size for infinite sets. According to that concept, a set A is smaller than a set B if, no matter how the ele-

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**ADELE PORKERT and Gödel were an unlikely but devoted couple. This photograph, taken at an outdoor Viennese cafe, is from the period of their long courtship. Porkert shielded Gödel from the worst of his irrational fears and was often the only person who could persuade him to eat. More than anyone else, she was responsible for keeping him alive and productive.**

ments of A are correlated in a one-to-one fashion with elements of B, some elements of B are always left over. Using this concept, Cantor proved that the set of natural numbers is smaller than the set of all decimal numbers. He further conjectured that no set has a size intermediate between those two—an assertion that came to be known as the continuum hypothesis.

In 1908 Cantor's compatriot, Ernst Zermelo, formulated a list of axioms for set theory. Among them was the axiom of choice, which states (in one version) that given any infinite collection of nonoverlapping sets, each of which contains at least one element, there is a set that contains exactly one element from each set in the collection. Though seemingly unobjectionable—why shouldn't one be able to select one element from each set?—the axiom of choice has a multitude of highly counterintuitive consequences. It implies, for example, that a sphere may be decomposed into a finite number of pieces that can be separated and reassembled, using only rigid motions, to form a new sphere having twice the volume of the first.

As a result, the axiom became highly controversial. Mathematicians suspected—correctly, as it turned out—that neither the axiom of choice nor the continuum hypothesis could be deduced from the other axioms of set theory. They feared that the use of those theorems in proofs might lead to contradictions. Gödel, however, proved that both principles are consistent with the other axioms.

Gödel's set-theoretic results answered a question that Hilbert had posed in 1900 in an address to the International Congress of Mathematicians. As such, they were a major achievement, but they were still not enough to earn him a permanent academic position. During his year at the Institute for Advanced Study and Notre Dame, his authorization to teach at Austrian universities lapsed. When he returned to Vienna to be reunited with his wife in the summer of 1939, he was summoned for a military physical and declared fit for service in the Nazi armed forces.

## Undecidable Propositions

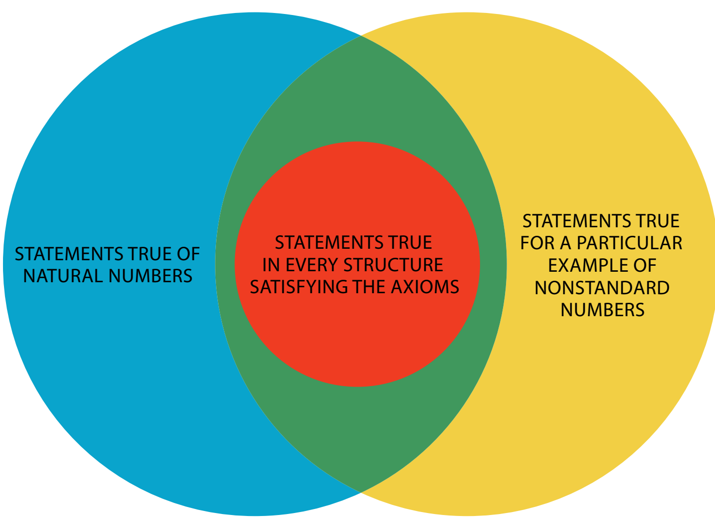
Gödel's most famous contribution was the proof that some statements about natural numbers are true but unprovable. Unfortunately, a long history of attempts to find statements that are undecidable—that is, neither provable nor disprovable—has led to few simple examples. One is the following sentence:

This statement is unprovable.

The above can be coded as a numerical equation according to a formula devised by Gödel. The equation is not provable and therefore affirms the meaning of the English-language proposition. That means, however, that the statement is true.

A less trivial example involves polynomial equations. One can state, for example, that a certain polynomial equation has no roots (that is, solutions) that are whole numbers. Such statements can turn out to be undecidable.

Gödel's proof demonstrated that the axioms of number theory are incomplete. That is, there are true statements about the natural numbers that cannot be proved by those axioms. His argument implies that "nonstandard numbers"—entities that obey the said axioms but have some properties that are different from those of natural numbers—exist. Because everything proved from axioms (*red*) must apply to all entities that obey the axioms, some true statements about natural numbers (*blue, green* and *red*) must be unprovable (*blue* and *green*). *—J.W.D.*




BRYAN CHRISTIE

### Deepening Fears

Until then, Gödel appears to have been aloof to the frightening developments in Europe. He was interested in politics and kept abreast of events but remained oddly unmoved by them. His lack of emotional engagement with people may have kept him from appreciating the significance of what was happening. He seemed oblivious to the fates befalling his colleagues and professors, many of whom were Jewish, and stayed immersed in his work while the world around him fell apart. Finally, he realized it was taking him down as well.

In that desperate situation, unemployed and subject to imminent induction, he enlisted the support of the Institute for Advanced Study to help secure exit visas for himself and his wife. His efforts were successful, and in January 1940 the two of them began a long journey eastward via the trans-Siberian railway. From Yokohama they continued by ship to San Francisco and from there by train to Princeton, where they arrived around the middle of March.

Gödel never again left the U.S. After a series of annual appointments he was made a permanent member of the institute faculty in 1946. Two years later he acquired American citizenship. (On that occasion, the judge who swore him in made the unfortunate error of asking his opinion of the U.S. Constitution and unleashed a pent-up lecture on its inconsistencies.) But Gödel was not promoted to professor until 1953—the same year he was elected to membership in the National Academy of Sciences—in part because his expressed fear that poison gases were escaping from his refrigerator aroused continuing concern about his mental stability. During those years, his friend Albert Einstein took it on himself to look after Gödel as best he could, taking a walk with him every day. Their conversations seem to have had a calming effect on Gödel.

After his emigration Gödel gave up work in set theory and turned to philosophy and relativity theory. In 1949 he demonstrated that universes in which time travel into the past is possible were compatible with Einstein's equations. In 1950 he spoke about those results at the International Congress of Mathematicians, and the next year he delivered the prestigious Gibbs Lecture to the annual meeting of the American Mathematical Society. But in the interval between those addresses he nearly succumbed to a bleeding ulcer, neglected until an extremely advanced stage because of his distrust of doctors.

Gödel's last published paper appeared in 1958. After that he withdrew more and more into himself, becoming increasingly emaciated, paranoid and hypochondriacal. He last appeared in public in 1972, when the Rockefeller University granted him an honorary doctorate. Three years later he was awarded the National Medal of Science but declined to attend the awards ceremony on grounds of ill health.

On July 1, 1976, having reached the mandatory retirement age of 70, Gödel became professor emeritus at the institute. His responsibilities did not lessen, though, because his wife, who for so many years had nurtured and protected him, had suffered an incapacitating stroke a few months before. It was his turn to care for her. He did so, devotedly, until July 1977, when she underwent emergency surgery and was hospitalized for nearly six months.

At about that time Morgenstern, the friend who had helped to look after Gödel in the years after Einstein's death in 1955, died of cancer. Gödel was thus left to fend for himself against his growing paranoia. In the face of that, he declined rapidly. His fear of poisoning led to self-starvation, from which he died on January 14, 1978.

COURTESY OF RICHARD ARENS

**WALKS WITH ALBERT EINSTEIN on the grounds of the Institute for Advanced Study were part of the routine that kept Gödel functioning. This photograph is from 1954.**

Adele Gödel survived her husband by three years. At her death, on February 4, 1981, she bequeathed rights to Gödel's papers to the Institute for Advanced Study. Although an outcast in Princeton's snobbish society, she was proud of her husband's work and probably realized that he would not have accomplished much had she not kept him functioning.

Gödel published remarkably few papers during his lifetime—fewer, indeed, than any other great mathematician except Bernhard Riemann—but their impact has been enormous. They have affected virtually every branch of modern logic. During the past decade, other papers of his have been translated from the obsolete German shorthand he used and published posthumously in the third volume of his *Collected Works*. Their contents, including his formalization of the so-called ontological argument for the existence of God, have begun to attract attention as well. At last, the breadth of his work is becoming known to those outside the mathematical community. SA

*The Author*

JOHN W. DAWSON, JR., catalogued Kurt Gödel's papers at the Institute for Advanced Study in Princeton, N.J. He has served as co-editor of Gödel's *Collected Works* since the inception of that project. He received his doctorate in mathematical logic from the University of Michigan in 1972 and is a professor of mathematics at Pennsylvania State University at York. He has a particular interest in axiomatic set theory and the history of logic.

*Further Reading*

GÖDEL'S PROOF. Ernst Nagel and James R. Newman. New York University Press, 1958.

COLLECTED WORKS, Vols. 1–3. Kurt Gödel. Edited by Solomon Feferman et al. Oxford University Press, 1986, 1990, 1995.

LOGICAL DILEMMAS: THE LIFE AND WORK OF KURT GÖDEL. John W. Dawson, Jr. A. K. Peters Ltd., Wellesley, Mass., 1997.

GÖDEL, ESCHER, BACH: AN ETERNAL GOLDEN BRAID. Douglas R. Hofstadter. (Originally published by Basic Books, 1979.) Basic Books, 1999.