Resolving Zeno's Paradoxes

For millennia, mathematicians and philosophers have tried to refute Zeno's paradoxes, a set of riddles suggesting that motion is inherently impossible. At last, a solution has been found

by William I. McLaughlin

Once upon a time Achilles met a tortoise in the road. The tortoise, whose mind was quicker than his feet, challenged the swift hero to a race. Amused, Achilles accepted. The tortoise asked if he might have a head start, as he was truly much slower than the demigod. Achilles agreed happily, and so the tortoise started off. After taking quite a bit of time to fasten one of his sandal's ankle straps, Achilles bolted from the starting line. In no time at all, he ran half the distance that separated him from the tortoise. Within another blink, he had covered three quarters of the stretch. In another instant, he made up seven eighths and in another, fifteen sixteenths. But no matter how fast he ran, a fraction of the distance remained. In fact, it appeared that the hero could never overtake the plodding tortoise.

Had Achilles spent less time in the gym and more time studying philosophy, he would have known that he was acting out the classic example used to illustrate one of Zeno's paradoxes, which argue against the possibility of all motion. Zeno designed the paradox of Achilles and the tortoise, and its companion conundra (more about them later), to support the philosophical theories of his teacher, Parmenides.

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Both men were citizens of the Greek colony of Elea in southern Italy. In approximately 445 B.C., Parmenides and Zeno met with Socrates in Athens to exchange ideas on basic philosophical issues. The event, one of the greatest recorded intellectual encounters (if it really took place), is commemorated in Plato's dialogue *Parmenides.* Parmenides, a distinguished thinker nearly 65 years old, presented to the young Socrates a startling thesis: "reality" is an unchanging single entity, seamless in its unity. The physical world, he argued, is monolithic. In particular, motion is not possible. Although the rejection of plurality and change appears idiosyncratic, it has, in general outline, proved attractive to numerous scholars. For example, the "absolute idealism" of the Oxford philosopher F. H. Bradley (1846–1924) has points in common with the Parmenidean outlook.

This portrayal of the world is contrary to our everyday experience and relegates our most fundamental perceptions to the realm of illusion. Parmenides relied on Zeno's powerful arguments, which were later recorded in the writings of Aristotle, to support his case. For two and a half millennia, Zeno's paradoxes have provoked debates and stimulated analyses. At last, using a formulation of calculus that was developed in just the past decade or so, it is possible to resolve Zeno's paradoxes. The resolution depends on the concept of infinitesimals, known since ancient times but until recently viewed by many thinkers with skepticism.

The tale of Achilles and the tortoise depicts one of Zeno's paradoxes, usually denoted "The Dichotomy": any distance, such as that between the two contenders, over which an object must traverse can be halved ($1/2$, $1/4$, $1/8$ and so on) into an infinite number of spatial segments, each representing some distance yet to be traveled. As a result, Zeno asserts that no motion can be completed because some distance, no matter how small, always remains. It is important to note that he does not say that infinitely many stretches cannot add up to a finite distance (glancing at the geometry of an infinitely partitioned line shows immediately, without any sophisticated calculations, that an infinite number of pieces sum to a finite interval). Rather the force of Zeno's objection to the idea of motion comes from the obligation to explain how an infinite number of acts—crossing one interval—can be serially completed.

Zeno made a second attack on the conceptual underpinnings of motion by viewing this first argument from a slightly different perspective. His second paradox is as follows: Before an object, say, an arrow, gets to the halfway mark of its supposed journey (an achievement granted in the preceding case), it must first travel a quarter of the distance. As in Zeno's first objection, this reasoning can be continued indefinitely to yield an infinite regress, thus leading to his insistence that motion could never be initiated.

Zeno's third paradox takes a different tack altogether. It asserts that the very concept of motion is empty of content. Zeno invites us to consider the arrow at any one instant of its flight. At this point in time, the arrow occupies a region of space equal to its length, and no motion whatsoever is evident. Because this observation is true at every instant, the arrow is never in motion. This objection, in a historical sense, proved the most troublesome for would-be explainers of Zeno's paradoxes.

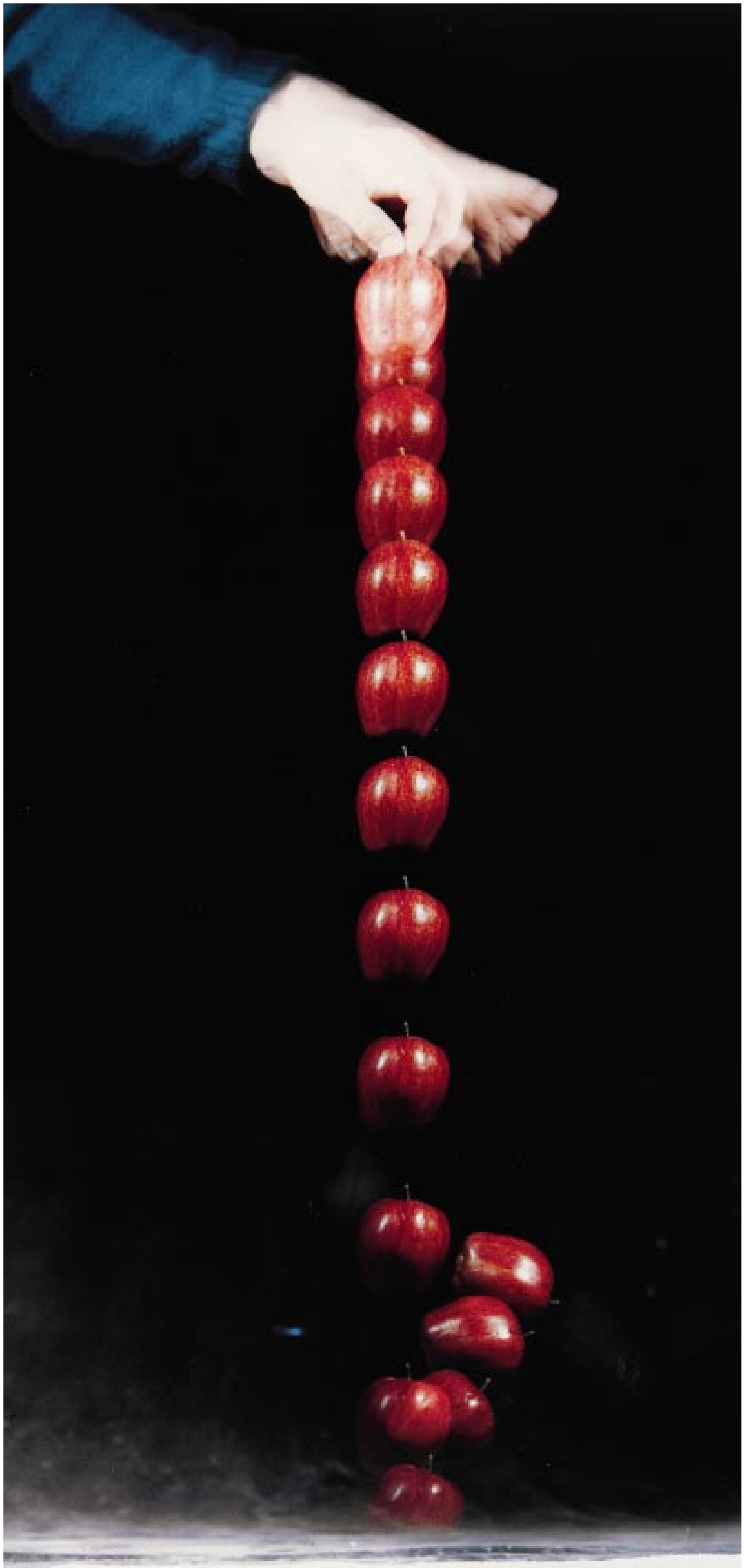
Many philosophers and mathematicians have made various attempts to answer Zeno's objections. The most direct approach has simply been to deny that a problem exists. For example, Johann Gottlieb Waldin, a German professor of philosophy, wrote in 1782 that the Eleatic, in arguing against motion, assumed that motion exists. Evidently the good professor was not acquainted with the form of argument known as reductio ad absurdum: assume a state of affairs and then show that it leads to an illogical conclusion.

Nevertheless, other scholars made progress by wrestling with how an infinite number of actions might occur in the physical world. Their explanations have continually been intertwined with the idea of an infinitesimal, an interval of space or time that embodies the quintessence of smallness. An infinitesimal quantity, some surmised, would be so very near zero as to be numerically impotent; such quantities would elude all measurement, no matter how precise, like sand through a sieve.

Giovanni Benedetti (1530–1590), a predecessor of Galileo, postulated that when an object appeared to be frozen in midair to Zeno, he was in fact seeing only part of the action, as though one were watching a slide show instead of a movie. Between the static images Zeno saw were infinitesimally small instants of time in which the object moved by equally small distances.

Others sidestepped the issue by arguing that intervals in the physical world cannot simply be subdivided an infinite number of times. Friedrich Adolf Trendelenburg (1802–1872) of the University of Berlin built an entire philosophical system that explained human perceptions in terms of motion. In doing so, he freed himself from explaining motion itself.

Similarly, in this century, the English philosopher and mathematician Alfred North Whitehead (1861–1947) constructed a system of metaphysics based on change, in which motion was a special case. Whitehead responded to Zeno's objections by insisting that events in the physical world had to have some extent; namely, they could not be pointlike. Likewise, the Scottish philosopher David Hume (1711–1776) wrote, "All the ideas of quantity upon which mathematicians reason, are nothing but particular, and such as are suggested by the senses and imagination



FALLING APPLE? Zeno would argue that because the apple appears to be frozen in midair at each instant of its supposed descent, it is never in motion. Moreover, Zeno would assert that there is no proof that the apple will ever reach the ground. Before it arrives there, it must first fall half of the distance between the man's hand and the ground. After that, it must fall half of the remaining distance and half of that again, and so on. How can it be that some fractional distance does not always remain between the apple and the ground? Using similar logic, Zeno would question whether an apple can even begin to fall.

RACE between Achilles and the tortoise illustrates one of Zeno's paradoxes. Achilles gives the tortoise a head start. He must then make up half the distance between them, then three fourths, then seven eighths and so on, ad infinitum. In this way, it would seem he could never come abreast of the sluggish animal.

and consequently, cannot be infinitely divisible."

Either way, the subject of infinitesimals (and whether they exist or not) generated a long and acrimonious literature of its own. Until recently, most mathematicians thought them to be a chimera. The Irish bishop George Berkeley (1685–1753) is noted principally for his idealistic theory, which denied the reality of matter, but he, too, wrestled with infinitesimals. He believed them ill conceived by the mathematicians of the time, including Newton. "They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?" He observed further: "Whatever mathematicians may think of fluxions [rates of change], or the differential calculus, and the like, a little reflexion will shew them that, in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense."

Indeed, mathematicians found infinitesimals hard to skirt in the course of their discoveries, no matter how distasteful they found them in theory. Some historians believe the great Archimedes (circa 287–212 B.C.) achieved some of his mathematical results using infinitesimals but employed more conventional modes for public presentations. Infinitesimals left their mark during the 17th and 18th centuries as well in the development of differential and integral calculus. Elementary textbooks have long appealed to "practical infinitesimals" to convey certain ideas in calculus to students.

When analysts thought about rigorously justifying the existence of these small quantities, innumerable difficulties arose. Eventually, mathematicians of the 19th century invented a technical substitute for infinitesimals: the so-called theory of limits. So complete was its triumph that some mathematicians spoke of the "banishment" of infinitesimals from their discipline. By the 1960s, though, the ghostly tread of infinitesimals in the corridors of mathematics became quite real once more, thanks to the work of the logician Abraham Robinson of Yale University [see "Nonstandard Analysis," by Martin Davis and Reuben Hersh; SCIENTIFIC AMERICAN, June 1972]. Since then, several methods in addition to Robinson's approach have been devised that make use of infinitesimals.

When my colleague Sylvia Miller and I started our work on Zeno's paradoxes, we had the advantage that infinitesimals had become mathematically respectable. We were intuitively drawn to these objects because they seem to provide a microscopic view of the details of motion. Edward Nelson of Princeton University created the tool we found most valuable in our attack, a brand of nonstandard analysis known by the rather arid name of internal set theory (IST). Nelson's method produces startling interpretations of seemingly familiar mathematical structures. The results are similar, in their strangeness, to the structures of quantum theory and general relativity in physics. Because these two theories have taken the better part of a century to gain widespread acceptance, we can only admire the power of Nelson's imagination.

Nelson adopted a novel means of defining infinitesimals. Mathematicians typically expand existing number systems by tacking on objects that have desirable properties, much in the same way that fractions were sprinkled between the integers. Indeed, the number system employed in modern mathematics, like a coral reef, grew by accretion onto a supporting base: "God made the integers, all the rest is the work of man," declared Leopold Kronecker (1823–1891). Instead the way of IST is to "stare" very hard at the existing number system and note that it already contains numbers that, quite reasonably, can be considered infinitesimals.

Technically, Nelson finds nonstandard numbers on the real line by adding three rules, or axioms, to the set of 10 or so statements supporting most mathematical systems. (Zermelo-Fraenkel set theory is one such foundation.) These additions introduce a new term, standard, and help us to determine which of our old friends in the number system are standard and which are nonstandard. Not surprisingly, the infinitesimals fall in the nonstandard category, along with some other numbers I will discuss later.

Nelson defines an infinitesimal as a number that lies between zero and every positive standard number. At first, this might not seem to convey any particular notion of smallness, but the standard numbers include every concrete number (and a few others) you could write on a piece of paper or generate in a computer: 10, pi, $^1/_{1000}$ and so on. Hence, an infinitesimal is greater than zero but less than any number, however small, you could ever conceive of writing. It is not immediately apparent that such infinitesimals do indeed exist, but the conceptual validity of IST has been demonstrated to a degree commensurate with our justified belief in other mathematical systems.

Still, infinitesimals are truly elusive

entities. Their elusiveness rests on the mathematical fact that two concrete numbers—those having numerical content—cannot differ by an infinitesimal amount. The proof, by reductio ad absurdum, is easy: the arithmetic difference between two concrete numbers must be concrete (and hence, standard). If this difference were infinitesimal, the definition of an infinitesimal as less than all standard numbers would be violated. The consequence of this fact is that both end points of an infinitesimal interval cannot be labeled using concrete numbers. Therefore, an infinitesimal interval can never be captured through measurement; infinitesimals remain forever beyond the range of observation.

So how can these phantom numbers be used to refute Zeno's paradoxes? From the above discussion it is clear that the points of space or time marked with concrete numbers are but isolated points. A trajectory and its associated time interval are in fact densely packed with infinitesimal regions. As a result, we can grant Zeno's third objection: the arrow's tip is caught "stroboscopically" at rest at concretely labeled points of time, but along the vast majority of the stretch, some kind of motion is taking place. This motion is immune from Zenonian criticism because it is postulated to occur inside infinitesimal segments. Their ineffability provides a kind of screen or filter.

Might the process of motion inside one of these intervals be a uniform advance across the interval or an instantaneous jump from one end to the other? Or could motion comprise a series of intermediate steps or else a process outside of time and space altogether? The possibilities are infinite, and none can be verified or ruled out since an infinitesimal interval can never be monitored. Credit for this rebuttal is due to Benedetti, Trendelenburg and Whitehead for their earlier insights, which can now be formalized by means of IST.

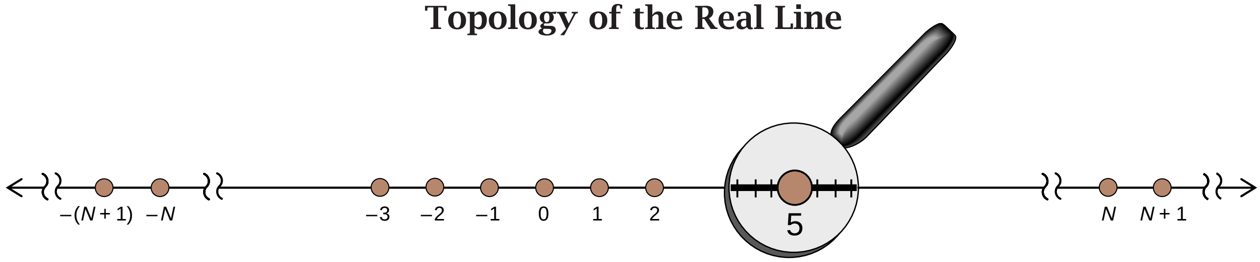
We can answer Zeno's first two objections more easily than we did the third, but we need to use another mathematical fact from IST. Every infinite set of numbers contains a nonstandard number. Before drawing out the Zenonian implications of this statement, it is necessary to talk about the two other types of nonstandard numbers that are readily manufactured from infinitesimal numbers. First, take all the infinitesimals, which by definition are wedged between zero and all the positive, standard numbers, and put a minus sign in front of each one. Now there is a symmetrical clustering of these small objects about zero. To create "mixed" nonstandard numbers, take any standard number, say, one half, and add to it each of the nonstandard infinitesimals in the grouping around zero. This act of addition translates the original cluster of infinitesimals to positions on either side of one half. Similarly, every standard number can be viewed as having its own collection of nearby, nonstandard numbers, each one only an infinitesimal distance from the standard number.

The third type of nonstandard number is simply the inverse of an infinitesimal. Because an infinitesimal is very small, its inverse will be very large (in the standard realm, the inverse of one millionth is one million). This type of nonstandard number is called an unlimited number. The unlimited numbers, though large, are finite and hence smaller than the truly infinite numbers created in mathematics. These unlimited numbers live in a kind of twilight zone between the familiar standard numbers, which are finite, and the infinite ones.

If, as demonstrated in IST, every infinite set contains a nonstandard number, then the infinite series of checkpoints Zeno used to gauge motion in his first argument must contain a mixed, nonstandard number. In fact, as Zeno's infinite series of numbers creeps closer to one, a member of that series will eventually be within an infinitesimal distance from one. At that point, all succeeding members of the series will be nonstandard members of the cluster about one, and neither Zeno nor anyone else will be able to chart the progress of a moving object in this inaccessible region.

There is an element of irony in using infinity, Zeno's putative weapon, to deflate his claims. To refute Zeno's first paradox, we need only state the epistemological principle that we are not responsible for explaining situations we cannot observe. Zeno's infinite series of checkpoints contains nonstandard numbers, which have no numerical meaning, and so we reject his argument based on these entities. Because no one could ever, even in principle, observe the full domain of check-

Topology of the Real Line



The real numbers consist of the integers (positive and negative whole numbers), rational numbers (those that can be expressed as a fraction) and irrational numbers (those that cannot be expressed as a fraction). The real numbers can be represented as points on a straight line known as the real line (*above*).

The mathematician Edward Nelson of Princeton University labeled three types of numbers as nonstandard within this standard number system. Infinitesimal nonstandard numbers are smaller than any positive standard number yet are greater than zero. Mixed nonstandard numbers, shown grouped around the integer 5, result from adding and subtracting infinitesimal amounts to standard numbers. In fact, every standard number is surrounded by such mixed, nonstandard neighbors. Unlimited nonstandard numbers, represented as N and $N + 1$, are the inverses of infinitesimal nonstandard numbers. Each unlimited number is greater than every standard number and yet less than the infinite real numbers. The nonstandard real numbers prove useful in resolving Zeno's paradoxes.

points that his objection addresses, the objectionable behavior he postulates for the moving object is moot. Many descriptions of motion in the microrealm other than that containing the full series of checkpoints could apply, and just because his particular scenario causes conceptual problems, there is no reason to anathematize the idea of motion. His second argument, attempting to show that an object can never even start to move, suffers from the same malady as the first, and we reject it on like grounds.

We have resolved Zeno's three paradoxes using some technical results from IST and the principle that nonstandard numbers are not suitable for describing matters of fact, observed or purported. Still, more can be said regarding the matter than just the assurance that Zeno's objections do not preclude motion. Indeed, we can construct a theory of motion using a very powerful result from IST. The theory yields the same results as do the tools of the calculus, and yet it is easier to visualize and does not fall prey to Zeno's objections.

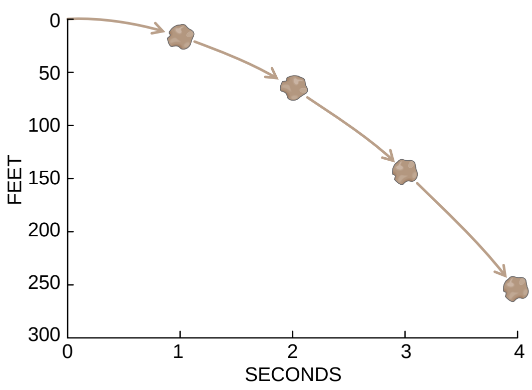
A theorem proved in IST states that there exists a finite set, call it F, that contains all the standard numbers! The corollary that there are only a finite number of standard numbers would seem to be true, but surprisingly, it is not. In developing IST, Nelson needed to finesse the conventional way mathematicians form objects. A statement in IST is called internal if it does not contain the label "standard." Otherwise, the statement is called external. Mathematicians frequently create subsets from larger sets by predicating a quality that characterizes each of the objects in the subset—the balls that are red or the integers that are even. In IST, however, it is forbidden to use external predicates, such as standard, to define subsets; the stricture is introduced to avoid contradictions. For example, imagine the set of all standard numbers in F. This set would be finite because it is a subset of a finite set. It would therefore have a least member, say, r. But then $r - 1$ would be a standard number less than r, when r was supposed to be the smallest standard number. Thus, we cannot say the standard numbers are finite or infinite in extent, because we cannot form the set of them and count them.

Nevertheless, the finite set F, though constrained as to how it can be visualized, is useful for constructing our theory of motion. This theory can be expressed quite simply as stepping through F, where each member of F represents a distinct moment. For convenience, consider only those members of F that fall between 0 and 1. Let time 0 be the instant when we start tracking a moving object. The second instant when we might try to observe the object is at time f_1, where f_1 is the smallest member of F that is greater than 0. Ascending through F in this fashion, we eventually reach time f_n, where f_n is the largest member of F less than 1. In one more step, we reach 1 itself, the destination in this example. In order to walk through a noninfinitesimal distance, such as the span from 0 to 1 using infinitesimal steps, the subscript n of f_n must be an unlimited integer. The process of motion then is divided into $n+1$ acts, and because $n+1$ is also finite, this number of acts can be completed sequentially.

Of the possible observing times identified earlier, the object's progress could be reported solely at those instants corresponding to certain standard numbers in F. (By the way, f_1 and f_n would be nonstandard, as they are infinitesimally close to 0 and 1, respectively.) For example, although we can express a standard number to any finite (but not unlimited) number of decimal places and use this approximation as a measurement label, we cannot access the unlimited tail of the expansion to alter a digit and thus define a nonstandard,

Calculus by Means of Infinitesimals

To see the relation between infinitesimals and differential calculus, consider the simple case of a falling stone. The distance the stone has traveled in feet can be calculated from the formula $s = 16t^2$, where t equals the time elapsed in seconds. For example, if a stone has fallen for two seconds, it will have traveled 64 feet.



Suppose, however, one wishes to calculate the instantaneous velocity of the stone. The average speed of a moving object equals the total distance it travels divided by the total amount of time it takes. By using this formula over an infinitesimal change in the total distance and time, one can calculate a fair approximation of an object's instantaneous velocity.

Let dt represent an infinitesimal change in time and ds an infinitesimal change in distance. The computation for the velocity of the stone after one second of travel, then, will be as follows: The time frame under consideration ranges from $t = 1$ to $t = 1 + dt$. The position of the stone during that time changes from $s = 16(1)^2$ to $s = 16(1 + dt)^2$. The total change in distance, $32dt + 16dt^2$, divided by dt, is the desired average velocity, $32 + 16dt$.

Because $16dt$ is but an infinitesimal amount, undetectable for all intents and purposes, it can be considered equal to 0. Thus, after one second of travel, the formula yields the stone's instantaneous velocity as 32 feet per second.

This manipulation, of course, resembles those used in traditional, differential calculus. There the small residue $16dt$ cannot be dropped at the end of the calculation; it is a noninfinitesimal quantity. Instead, in this calculus, it must be argued away using the theory of limits. In essence, the limit process renders the interval of length dt sufficiently small so that the average velocity is arbitrarily close to 32. As before, the instantaneous velocity of the stone after one second of travel equals 32 feet per second. Similarly, judicious use of infinitesimal regions facilitates the computation of the area of complicated regions, a basic problem of integral calculus. Some think the newer calculus is pedagogically superior to calculus without infinitesimals. Nevertheless, both methods are equally rigorous and yield identical results.

***THE MEASURERS,* a 17th-century Dutch painting attributed to Hendrik van Balen, illustrates the words of the Roman poet Horace: "There is measure in all things." No matter how precise measurements become, however, infinitesimal amounts will forever escape our grasp, since any useful unit of measure must correspond to some standard number.**

infinitesimally close neighbor. Only concrete standard numbers are effective as measurement labels; the utility of their nonstandard neighbors for measurement is illusory.

Much is superfluous in this theory of motion, and much is left unsaid. It suffices, however, in the sense that it can easily be translated into the symbolic notation of the integral or differential calculus, commonly used to describe the details of motion [*see box on opposite page*]. More important in the present context, the finiteness of the set F enables us to jump over the pitfalls in Zeno's first two paradoxes. His third objection is dodged as before: motion in real time is an unknown process that takes place in infinitesimal intervals between the standard points of F; the nonstandard points of F are irrelevant given that they cannot be observed.

For many centuries, Zeno's logic stood mostly intact, proving the refractory nature of his arguments. A resolution was made possible through two basic features of IST: first, the ability to partition an interval of time or space into a finite number of ineffable infinitesimals and, second, the fact that standardly labeled points—the only ones that can be used for measurement—are isolated objects on the real line. Is our work merely the solution to an ancient puzzle? Possibly, but there are several directions in which it might prove extensible.

Aside from its mathematical value, IST is ripe with epistemological import, as this analysis has shown. It might well be modified to constitute a general epistemic logic. Also, infinitesimal intervals, or their generalization, would promise a technical resource to house Whitehead's so-called actual entities, the generative atoms of his philosophical system. Finally, the current theory of motion and the predictions of quantum physics are not dissimilar in that they both restrict the observation of certain events to discrete values. Of course, this theory of motion is not a version of quantum mechanics (nor relativity theory, for that matter). Because the theory resulted from a thought experiment on Zeno's terms, it holds no direct connection to present physical theory. Moreover, the specific rules inherited from IST are probably not those best suited to describe reality. Modern physics might adapt the IST approach by modifying its rule system and introducing "physical constants," perhaps by assigning parameters to the set F.

But maybe not. Still, the simplicity and elegance of such thought experiments have catalyzed research throughout the ages. Notable examples include Heinrich W. M. Olbers, questioning why the sky is dark at night despite stars in every direction, or James Clerk Maxwell, summoning a meddling, microscopic demon to batter the second law of thermodynamics. Likewise, Zeno's arguments have stimulated examinations of our ideas about motion, time and space. The path to their resolution has been eventful.

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