**HISTORY OF LOGIC**

* Euclid (fl 300BC) - 23 definitions, 5 postulates, 5 common notions, 48 propositions
  + yet Greeks were wary of the concept of **infinity** \therefore \!\,parallel postulate
* Plato: mathematics is discovered
* Greek heritage: trisect angle, square the circle, double the cube: 2300 years to solve
* Zeno - paradox - **infinitesimals** - tortoise and the hare – obvious (to Brophy) but unsolved?
* Aristotle (fl 350BC) - formal logic system; **Kant - said in 1787**, "Aristotle's system complete and cannot be improved."
* Aquinas (fk1250): 5 proofs of God; use of infinite regressions
* Descartes (fl 1637) - combined algebra and geometry: circle at origin = x2 + y2 = r2

“cogito ergo sum.”

* Newton fl 1670- **infinitesimals** and God; calculus, alchemy, voted against marine chronometer. See book "Longitude"
* Gauss fl 1800, Bolyai, Riemann, Lobachevski; non-Euclidean geometries;

replaced the parallel postulate.

* Boole - 1847 –first algebraic notation for logic: AND (&, ^); OR (V)
* Hilbert (fl 1900) - set goals for foundations of logic, simple and **finitistic**
* Cantor (fl 1880) – set theory; many degrees of **infinity**; antinomies
* Peano (fl 1890) – Peano’s axioms for arithmetic
* Russell (fl 1910) - rules of inference; substitution, modus ponens, modus tolens; paradoxes
* Gödel (fl 1938) - truth is greater than proof; system of arithmetic is incomplete
* Fermat's Last Theorem, (Wile 1994); Marilyn vos Savant controversy - legitimacy of crossing math disciplines.
* Current: peer review, consensus, political correctness; “**what not why**”

**EUCLID:**

Familiar with Asymptotes, uncomfortable with parallel lines at infinity

The whole is greater than its parts. [not so in set theory]

Mathematicians lacked rules of inference for 2000 years

e.g. the rule of Substitution; the rule of Detachment

**NON-EUCLUDIAN GEOMETRIES**

Gauss; aware of consistency of non-Euclidean geometries, but did not publish – sensitive about reputation

Bolyai and Lobachevski, independently discovered Parabolic geometry (parallel lines meet)

Riemann discovered Hyperbolic geometry: suitable for Einstein's curved space

**HILBERT**

Hilbert sought to construct “absolute” proofs;

Prove without reference to another system.

Completely formalize the deductive system.

Drain the system of all meaning

Just have signs which can be combined or manipulated by a precise set of rules.

Important not to use any unavowed principles of reasoning.

2 + 3 = 5 contains only elementary signs

“2 +3 = 5" is an arithmetical formula in meta-mathematics.

An absolute proof achieves its objective by using a minimum of principles of inference

Does not assume the consistency of some other set of axioms.

Needs to show by a finitistic meta-math procedure the following: Two contradictory formulas, such a ‘0=0’ and ‘~0=0' cannot both be derived by stated rules of inference, i.e. the axioms.

**RUSSELL**:

Russell: pure math is the subject in which we do not know what we are talking about, or whether what are saying is true.

3 Volumes: Principia Mathematica. Third volume gets on to 1+1=2

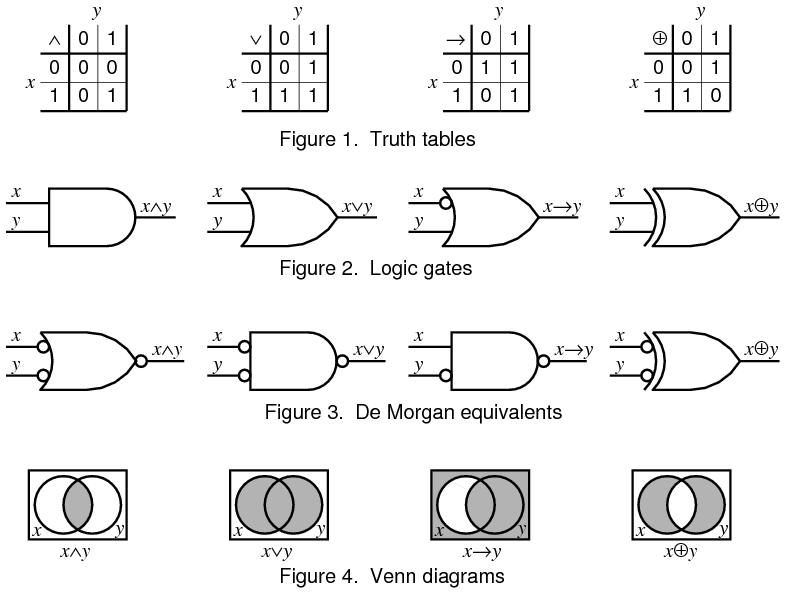
FAMOUS PARADOX: The set N of Normal Classes is a member of itself because it is Normal. But if it contains itself it is non-normal. So N is normal, if and only if it is not normal.

From Wikipedia: Let us call a set "abnormal" if it is a member of itself, and "normal" otherwise. For example, take the set of all squares. That set is not itself a square, and therefore is not a member of the set of all squares. So it is "normal". On the other hand, if we take the complementary set that contains all non-squares, that set is itself not a square and so should be one of its own members. It is "abnormal".

Now we consider the set of all normal sets, *R*. Attempting to determine whether *R* is normal or abnormal is impossible: If *R* were a normal set, it would be contained in the set of normal sets (itself), and therefore be abnormal; and if it were abnormal, it would not be contained in the set of normal sets (itself), and therefore be normal. This leads to the conclusion that *R* is both normal and abnormal: Russell's paradox.

**BOOLE:**

1847 George Boole: The Mathematical Analysis of Logic



**PEANO**

1. Zero is a number.

2. If A is a number, the successor of A is a number.

3. [zero](http://mathworld.wolfram.com/Zero.html) is not the successor of a number.

4. Two numbers of which the successors are equal are themselves equal.

5. ([induction axiom](http://mathworld.wolfram.com/InductionAxiom.html).) If a set S of numbers contains [zero](http://mathworld.wolfram.com/Zero.html) and also the successor of every number in S, then every number is in S

**GODEL’s Proof:**

**First theorem: [S cannot be both consistent and complete]**  Gödel's proof in a nutshell is to create a wff [well formed formula – [see wikipedia](http://en.wikipedia.org/wiki/Well-formed_formula)] that says in one interpretation, "This wff cannot be proved in S", then to prove that it is undecidable in S, and thereby to prove that it is true. In one interpretation this wff will say something about numbers; in the alternative interpretation used in the proof, it makes an assertion about its own provability. This crucial wff is usually called G, for Gödel.

[in English: It is impossible to give a meta mathematical proof of the consistency of a system comprehensive enough to contain the whole of arithmetic--unless more complicated rules of inference beyond the transformation rules. Also the proof must be finitistic.

**Second Theorem**: [**S which contains T is consistent if and only if it is inconsistent.**] Roughly, it says that no system of a certain strength can prove its own consistency unless it is inconsistent. It applies to the same respectable and well-made systems of arithmetic as his first theorem. The wff which asserts that the system is consistent can be proved to be unprovable in the system, on the assumption that the system is consistent.

[in English: Given any set of arithmetical statements, there are true statements that cannot be derived from the axioms. ]

PLEASE NOTE: It does not mean that there are ineluctable limits to human reasoning. Informal meta-mathematical reasoning may apply, but at great risk.

The proof is inscrutable: This site is better than most:

<http://www.earlham.edu/~peters/courses/logsys/g-proof.htm>

**GODEL NUMBERS & PROOF:**

1, Constant Signs: [~ not, 1], [v or, 2], [ \exists \!\, there exists, 4], [= equals, 5],

[0 zero, 6], [s successor, 7], [( left paren, 8], [) right paren, 9}, etc

2. Numerical Variables: [x, y, z = any numbers, assigned 11, 13 17, …]

3. Sentential Variables: <i.e. T or F.): [p, q, r, … any logical statement such as (\exists \!\,x)(xs=sy)

assigned Godel Numbers 112, 132, 172,….]

4. Predicate Variables: <Expressing Relationships>: [P, Q, R,…..e.g. Greater Than,

assigned Godel Numbers 113, 133, 173,….]

Applied to Formula: ( \exists \!\, x ) ( x = s y )

Replace with Godel Number: 8 4 11 9 8 11 5 7 13 9

Godel Number for Formula: 28  x 34 x 511  x 79 x 118  x 1311 x 175 x 197 x 2313 x 299

Every Godel Number is unique and has a unique set of factors\*\*. Godel goes on to show that (0 = 0) as well as its negation: ~(0 = 0) exist outside the arithmetization.

\*\*In [number theory](http://en.wikipedia.org/wiki/Number_theory) and [algebraic number theory](http://en.wikipedia.org/wiki/Algebraic_number_theory), the **Fundamental Theorem of Arithmetic** (or **Unique-Prime-Factorization Theorem**) states that any [integer](http://en.wikipedia.org/wiki/Integer) greater than 1 can be written as a *unique* product (up to ordering of the factors) of [prime numbers](http://en.wikipedia.org/wiki/Prime_number). For example,

6936 = 2^3 \times 3 \times 17^2 , \,\! this is not a Godel Number because it is missing the primes 5, 7, 11, 13

1200 = 2^4 \times 3 \times 5^2 . \,\!

100 = 62 + 82 = 22 x 52; since it doesn’t contain a prime factor = 3; it is not a Godel Number.

**THE FOLLOWING CONTAINS SOME NOTES ON**

**2. GODEL**

**3. EUCLIDS AXIOMS, inter alia**

**4. MODUS PONENS**

**5. MODUS TOLLENS**

**NOTES ON GODEL**

Gödel at age 25 in 1938 published in German on “Formally Undecidable Proposition of Principia Mathematica and Related Systems. The Principia Mathematica itself was a monumental 3 volume treatise on Logic and Foundations of Mathematics published 1910 by Whitehead and Russell.

In 1952, while being honored at Harvard for his most important work in math in modern times

few mathematicians, save Von Neumann, understand what Gödel was talking about

Geometry taught as a deductive system, and the axiomatic (postulates) method was accepted without proof and the derived theorems could be accepted as true with respect that they were the logical consequences of the axioms.

It was tacitly assumed that each sector of math could be supplied with a set of axioms sufficient for developing systematically the endless totality of true propositions about a given area of enquiry.

Gödel  showed these assumptions to be untenable.

Godel’s arithmetic method was somewhat analogous to Descarte’s algebraic method for geometry.

**Why is math so good;** It is not important whether the postulates assumed or the conclusion reached are true in the real world, but whether they are the natural logical consequences of the initial assumptions. In other words math is grounded in the structure of the statements, not the particular subject matter.

Therefore intuition is not a safe guide

Greg Cantor: p32, antinomies with infinite numbers.

Class either contains itself as a member or does not

Normal class does not contain itself

The class of mathematicians is normal

The class of thinkable things is not normal

in euclid’s proof of no greatest prime

Rule of inference: rule of substitution for sentential variables (can be either true or false).

mathematicians have been reasoning for 2000 years without being aware of all the principles underlying what they are doing.

if 5 > 3, then 5^2 >> 3^2

then (5+ ∞) > (3 + ∞)? NO!

system of notation; meaningless marks, strings

a formalized process is needed for absolute consistency

1. vocabulary: the signs to be used

2. formation rules: which combination of signs make formulas, i.e. sentences. the rules are grammar

3. transformation rules: rules of inference

4.axioms or primitives

transformation rules: (1) rule of substitution for sentential variables. if a sub is made, the same sub must be made for each occurrence. (2) rule of detachment (modus ponens). e.g. S(1) and if S(1) then S(2). it is always to deduce S(2)

The task is to show that at least one formula cannot be derived from the axioms. if (S1) and ~(S1) can be derived, then the system is not consistent.

to prove, need meta mathematical reasoning: structural property must satisfy 3 conditions

1. the property must be common to all axioms

2. the property must be hereditary under the transformation rule

3. the property must not belong to every formula that can be constructed with the formation rules of the system

there must find a property; like tautology; either it is raining or it is not raining. i.e. if joe is 200 pounds or joe is 200 pounds, then joe is 200 pound

summary:

every axiom of the system is a tautology

tautologyness is a hereditary property

every formula, theorem derived from the axioms is also a tautology

hence any formula that is not a tautology is not a theorem

one formula has been found (p or q) that is not a tautology

the formula is not a theorem

if axioms were inconsistent, every formula would be a theorem

therefore axioms are consistent

let us say that we can number all our expressions or definitions we want to use;

for example assume definition 17 is for: not divisible by any number other than 1 and itself. of course, deft 17 has the property designated by the expression.

assume definition 15 is : the product of some integer by itself.

15 has the property of being Richardian, and 17 has the property of being non-Richardian

n is Richardian, if and only if it is not Richardian. so that n is Richardian is both true and false. .

Gödel showed that meta-mathematical statements about a formalized mathematical calculus can indeed be represented by arithmetical formulas within the calculus. He devised a method by which a formula about a truth within the system, as well as a denial, could not be represented in the system; therefore the system was incomplete.

his proof is difficult

46 preliminary definitions

several theorems must be mastered before reaching a conclusion

Gödel numbers: for each sign , formula and proof

copy page 70 and 71

**EUCLIDS PROOF**

Definition 1.

A *point* is that which has no part.

[**Definition 2**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI2.html)**.**

A *line* is breadthless length.

[**Definition 3**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI3.html)**.**

The ends of a line are points.

[**Definition 4**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI4.html)**.**

A *straight line* is a line which lies evenly with the points on itself.

[**Definition 5**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI5.html)**.**

A *surface* is that which has length and breadth only.

[**Definition 6**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI6.html)**.**

The edges of a surface are lines.

[**Definition 7**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI7.html)**.**

A *plane surface* is a surface which lies evenly with the straight lines on itself.

[**Definition 8**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI8.html)**.**

A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

[**Definition 9**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI9.html)**.**

And when the lines containing the angle are straight, the angle is called *rectilinear.*

[**Definition 10**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI10.html)**.**

When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right,* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

[**Definition 11**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI11.html)**.**

An *obtuse angle* is an angle greater than a right angle.

[**Definition 12**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI11.html)**.**

An *acute angle* is an angle less than a right angle.

[**Definition 13**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI13.html)**.**

A *boundary* is that which is an extremity of anything.

[**Definition 14**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI13.html)**.**

A *figure* is that which is contained by any boundary or boundaries.

[**Definition 15**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI15.html)**.**

A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

[**Definition 16**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI15.html)**.**

And the point is called the *center* of the circle.

[**Definition 17**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI15.html)**.**

A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

[**Definition 18**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI15.html)**.**

A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

[**Definition 19**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI19.html)**.**

*Rectilinear figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

[**Definition 20**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI20.html)**.**

Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

[**Definition 21**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI20.html)**.**

Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

[**Definition 22**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI22.html)**.**

Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia.*

[**Definition 23**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI23.html)

*Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

**Postulates**

Let the following be postulated:

[**Postulate 1**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post1.html)**.**

To draw a straight line from any point to any point.

[**Postulate 2**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post2.html)**.**

To produce a finite straight line continuously in a straight line.

[**Postulate 3**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post3.html)**.**

To describe a circle with any center and radius.

[**Postulate 4**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post4.html)**.**

That all right angles equal one another.

[**Postulate 5**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post5.html)**.**

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

**Common Notions**

[**Common notion 1**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/cn.html)**.**

Things which equal the same thing also equal one another.

[**Common notion 2**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/cn.html)**.**

If equals are added to equals, then the wholes are equal.

[**Common notion 3**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/cn.html)**.**

If equals are subtracted from equals, then the remainders are equal.

[**Common notion 4**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/cn.html)**.**

Things which coincide with one another equal one another.

[**Common notion 5**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/cn.html)**.**

The whole is greater than the part.

**Propositions**

[**Proposition 1.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI1.html)

To construct an equilateral triangle on a given finite straight line.

[**Proposition 2.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI2.html)

To place a straight line equal to a given straight line with one end at a given point.

[**Proposition 3.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI3.html)

To cut off from the greater of two given unequal straight lines a straight line equal to the less.

[**Proposition 4.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI4.html)

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

[**Proposition 5.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI5.html)

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

[**Proposition 6.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI6.html)

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

[**Proposition 7.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI7.html)

Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

[**Proposition 8.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI8.html)

If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

[**Proposition 9.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI9.html)

To bisect a given rectilinear angle.

[**Proposition 10.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI10.html)

To bisect a given finite straight line.

[**Proposition 11.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI11.html)

To draw a straight line at right angles to a given straight line from a given point on it.

[**Proposition 12.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI12.html)

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

[**Proposition 13.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI13.html)

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

[**Proposition 14.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI14.html)

If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

[**Proposition 15.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI15.html)

If two straight lines cut one another, then they make the vertical angles equal to one another.

[**Corollary.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI15.html#cor) If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.

[**Proposition 16.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI16.html)

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

[**Proposition 17.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI17.html)

In any triangle the sum of any two angles is less than two right angles.

[**Proposition 18.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI18.html)

In any triangle the angle opposite the greater side is greater.

[**Proposition 19.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI19.html)

In any triangle the side opposite the greater angle is greater.

[**Proposition 20.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI20.html)

In any triangle the sum of any two sides is greater than the remaining one.

[**Proposition 21.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI21.html)

If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

[**Proposition 22.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI22.html)

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

[**Proposition 23.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI23.html)

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

[**Proposition 24.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI24.html)

If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

[**Proposition 25.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI25.html)

If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

[**Proposition 26.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI26.html)

If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

[**Proposition 27.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI27.html)

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

[**Proposition 28.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI28.html)

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

[**Proposition 29.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI29.html)

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

[**Proposition 30.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI30.html)

Straight lines parallel to the same straight line are also parallel to one another.

[**Proposition 31.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI31.html)

To draw a straight line through a given point parallel to a given straight line.

[**Proposition 32.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI32.html)

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

[**Proposition 33.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI33.html)

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

[**Proposition 34.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI34.html)

In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

[**Proposition 35.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI35.html)

Parallelograms which are on the same base and in the same parallels equal one another.

[**Proposition 36.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI36.html)

Parallelograms which are on equal bases and in the same parallels equal one another.

[**Proposition 37.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI37.html)

Triangles which are on the same base and in the same parallels equal one another.

[**Proposition 38.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI38.html)

Triangles which are on equal bases and in the same parallels equal one another.

[**Proposition 39.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI39.html)

Equal triangles which are on the same base and on the same side are also in the same parallels.

[**Proposition 40.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI40.html)

Equal triangles which are on equal bases and on the same side are also in the same parallels.

[**Proposition 41.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI41.html)

If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

[**Proposition 42.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI42.html)

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

[**Proposition 43.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI43.html)

In any parallelogram the complements of the parallelograms about the diameter equal one another.

[**Proposition 44.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI44.html)

To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

[**Proposition 45.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI45.html)

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

[**Proposition 46.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI46.html)

To describe a square on a given straight line.

[**Proposition 47.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI47.html)

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

[**Proposition 48.**](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI48.html)

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

**Guide**

**About the Definitions**

The *Elements* begins with a list of definitions. Some of these indicate little more than certain concepts will be discussed, such as [Def.I.1](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI1.html), [Def.I.2](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI2.html), and [Def.I.5](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI5.html), which introduce the terms point, line, and surface. (Note that for Euclid, the concept of line includes curved lines.) Others are substantial definitions which actually describe new concepts in terms of old ones. For example, [Def.I.10](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI10.html) defines a *right angle* as one of two equal adjacent angles made when one straight line meets another. Other definitions look like they're substantial, but actually are not. For instance, [Def.I.4](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI4.html) says a *straight line* "is a line which lies evenly with the points on itself." No where in the *Elements* is the defining phrase "which lies evenly with the points on itself" applicable. Thus, this definition indicates, at most, that some lines under discussion will be straight lines.

It has been suggested that the definitions were added to the *Elements* sometime after Euclid wrote them. Another possibility is that they are actually from a different work, perhaps older. In [Def.I.22](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI22.html) special kinds of quadrilaterals are defined including square, oblong (a rectangle that are not squares), rhombus (equilateral but not a square), and rhomboid (parallelogram but not a rhombus). Except for squares, these other shapes are not mentioned in the *Elements.* Euclid does use parallelograms, but they're not defined in this definition. Also, the exclusive nature of some of these terms—the part that indicates not a square—is contrary to Euclid's practice of accepting squares and rectangles as kinds of parallelograms.

**About the Postulates**

Following the list of definitions is a list of postulates. Each postulate is an axiom—which means a statement which is accepted without proof— specific to the subject matter, in this case, plane geometry. Most of them are constructions. For instance, [Post.I.1](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post1.html) says a straight line can be drawn between two points, and [Post.I.3](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post3.html) says a circle can be drawn given a specified point to be the center and another point to be on the circumference. The fourth postulate, [Post.I.4](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/post4.html), is not a constuction, but says that all right angles are equal.

**About magnitudes and the Common Notions**

The Common Notions are also axioms, but they refer to magnitudes of various kinds. The kind of magnitude that appears most frequently is that of straight line. Other important kinds are rectilinear angles and areas (plane figures). Later books include other kinds.

In proposition [III.16](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookIII/propIII16.html) (but nowhere else) angles with curved sides are compared with rectilinear angles which shows that rectilinear angles are to be considered as a special kind of plane angle. That agrees with Euclid's definition of them in [I.Def.9](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI9.html) and [I.Def.8](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/defI8.html).

Also in Book III, parts of circumferences of circles, that is, arcs, appear as magnitudes. Only arcs of equal circles can be compared or added, so arcs of equal circles comprise a kind of magnitude, while arcs of unequal circles are magnitudes of different kinds. These kinds are all different from straight lines. Whereas areas of figures are comparable, different kinds of curves are not.

Book V includes the general theory of ratios. No particular kind of magnitude is specified in that book. It may come as a surprise that ratios do not themselves form a kind of magnitude since they can be compared, but they cannot be added. See the guide on Book V for more information.

Number theory is treated in Books VII through IX. It could be considered that numbers form a kind of magnitude as pointed out by Aristotle. Beginning in Book XI, solids are considered, and they form the last kind of magnitude discussed in the *Elements.*

**The propositions**

Following the definitions, postulates, and common notions, there are 48 propositions. Each of these propositions includes a statement followed by a proof of the statement. Each statement of the proof is logically justified by a definition, postulate, common notion, or an earlier proposition that has already been proven. There are gaps in the logic of some of the proofs, and these are mentioned in the commentaries after the propositions. Also included in the proof is a diagram illustrating the proof.

Some of the propositions are constructions. A construction depends, ultimately, on the constructive postulates about drawing lines and circles. The first part of a proof for a constructive proposition is how to perform the construction. The rest of the proof (usually the longer part), shows that the proposed construction actually satisfies the goal of the proposition. In the list of propositions in each book, the constructions are displayed in red.

Most of the propositions, however, are not constructions. Their statements say that under certain conditions, certain other conditions logically follow. For example, [Prop.I.5](http://aleph0.clarku.edu/%7Edjoyce/java/elements/bookI/propI5.html) says that if a triangle has the property that two of its sides are equal, then it follows that the angles opposite these sides (called the "base angles") are also equal. Even the propositions that are not constructions may have constructions included in their proofs since auxiliary lines or circles may be needed in the explanation. But the bulk of the proof is, as for the constructive propositions, a sequence of statements that are logically justified and which culminates in the statement of the proposition.

# MODUS PONENS

# Modus ponens

### From Wikipedia, the free encyclopedia

In [classical logic](http://en.wikipedia.org/wiki/Classical_logic), **modus ponendo ponens** ([Latin](http://en.wikipedia.org/wiki/Latin_language) for *mode that affirms by affirming*;[[1]](http://en.wikipedia.org/wiki/Modus_ponens#cite_note-0) often abbreviated to **MP** or **modus ponens**) is a [valid](http://en.wikipedia.org/wiki/Valid), simple [argument form](http://en.wikipedia.org/wiki/Argument_form) sometimes referred to as **affirming the antecedent** or **the law of detachment**. It is closely related to another valid form of argument, [*modus tollens*](http://en.wikipedia.org/wiki/Modus_tollens) or "[denying the consequent](http://en.wikipedia.org/wiki/Denying_the_consequent)".

*Modus ponens* is a very common [rule of inference](http://en.wikipedia.org/wiki/Rule_of_inference), and takes the following form:

If *P*, then *Q*.

*P*.

Therefore, *Q*.[[2]](http://en.wikipedia.org/wiki/Modus_ponens#cite_note-1)

|  |
| --- |
|  |

## Formal notation

The *modus ponens* rule may be written in [sequent](http://en.wikipedia.org/wiki/Sequent) notation:

P \to Q, P \vdash Q

or in [rule form](http://en.wikipedia.org/wiki/Inference_rule):

\frac{P \rightarrow Q, P}{Q}.

The argument form has two premises. The first premise is the "if–then" or [conditional](http://en.wikipedia.org/wiki/Logical_conditional) claim, namely that P implies Q. The second premise is that P, the [antecedent](http://en.wikipedia.org/wiki/Antecedent_%28logic%29) of the conditional claim, is true. From these two premises it can be logically concluded that Q, the [consequent](http://en.wikipedia.org/wiki/Consequent) of the conditional claim, must be true as well. In [Artificial Intelligence](http://en.wikipedia.org/wiki/Artificial_Intelligence), modus ponens is often called [forward chaining](http://en.wikipedia.org/wiki/Forward_chaining).

An example of an argument that fits the form *modus ponens*:

If today is Tuesday, then I will go to work.

Today is Tuesday.

Therefore, I will go to work.

This argument is [valid](http://en.wikipedia.org/wiki/Validity), but this has no bearing on whether any of the statements in the argument are [true](http://en.wikipedia.org/wiki/Truth); for modus ponens to be a sound argument, the premises must be true for any true instances of the conclusion. An [argument](http://en.wikipedia.org/wiki/Logical_argument) can be valid but nonetheless [unsound](http://en.wikipedia.org/wiki/Soundness) if one or more premises are false; if an argument is valid *and* all the premises are true, then the argument is [sound](http://en.wikipedia.org/wiki/Soundness). For example, I might be going to work on Wednesday. In this case, the reasoning for my going to work (because it is Wednesday) is unsound. The argument is only sound on Tuesdays (when I go to work), but valid on every day of the week. A [propositional](http://en.wikipedia.org/wiki/Propositional_logic) argument using modus ponens is said to be [deductive](http://en.wikipedia.org/wiki/Deductive).

In single-conclusion [sequent calculi](http://en.wikipedia.org/wiki/Sequent_calculus), *modus ponens* is the Cut rule. The [cut-elimination theorem](http://en.wikipedia.org/wiki/Cut-elimination_theorem) for a calculus says that every proof involving Cut can be transformed (generally, by a constructive method) into a proof without Cut, and hence that Cut is [admissible](http://en.wikipedia.org/wiki/Admissible_rule).

The [Curry-Howard correspondence](http://en.wikipedia.org/wiki/Curry-Howard_correspondence) between proofs and programs relates modus ponens to [function application](http://en.wikipedia.org/wiki/Function_application): if *f* is a function of type *P → Q* and *x* is of type *P*, then *f x* is of type *Q*.

## Justification via truth table

The validity of *modus ponens* in classical two-valued logic can be clearly demonstrated by use of a [truth table](http://en.wikipedia.org/wiki/Truth_table).

|  |  |  |
| --- | --- | --- |
| **p** | **q** | **p → q** |
| **T** | **T** | **T** |
| **T** | **F** | **F** |
| **F** | **T** | **T** |
| **F** | **F** | **T** |

In instances of *modus ponens* we assume as premises that p → q is true and p is true. Only one line of the truth table - the first - satisfies these two conditions. On this line, q is also true. Therefore, whenever p → q is true and p is true, q must also be true.

MODUS TOLLENSIn [classical logic](http://en.wikipedia.org/wiki/Classical_logic), **modus tollens** (or **modus tollendo tollens**)[[1]](http://en.wikipedia.org/wiki/Modus_tollens#cite_note-0) ([Latin](http://en.wikipedia.org/wiki/Latin_language) for "the way that denies by denying")[[2]](http://en.wikipedia.org/wiki/Modus_tollens#cite_note-1) has the following [argument form](http://en.wikipedia.org/wiki/Argument_form):

If *P*, then *Q*.

*¬Q*

Therefore, *¬P*.[[3]](http://en.wikipedia.org/wiki/Modus_tollens#cite_note-2)

It can also be referred to as **denying the consequent**, and is a [valid](http://en.wikipedia.org/wiki/Validity) form of [argument](http://en.wikipedia.org/wiki/Argument) (unlike similarly-named but invalid arguments such as [*affirming the consequent*](http://en.wikipedia.org/wiki/Affirming_the_consequent) or [*denying the antecedent*](http://en.wikipedia.org/wiki/Denying_the_antecedent)). (See also [*modus ponens*](http://en.wikipedia.org/wiki/Modus_ponens) or "affirming the antecedent".)

Modus tollens is sometimes confused with [indirect proof](http://en.wikipedia.org/wiki/Indirect_proof) (assuming the negation of the proposition to be proved and showing that this leads to a [contradiction](http://en.wikipedia.org/wiki/Contradiction)) or [proof by contrapositive](http://en.wikipedia.org/wiki/Proof_by_contrapositive) (proving If P, then Q by a proof of the equivalent [contrapositive](http://en.wikipedia.org/wiki/Contrapositive) If not-Q, then not-P).

The validity of *modus tollens* can be clearly demonstrated through a [truth table](http://en.wikipedia.org/wiki/Truth_table).

|  |  |  |
| --- | --- | --- |
| **p** | **q** | **p → q** |
| **T** | **T** | **T** |
| **T** | **F** | **F** |
| **F** | **T** | **T** |
| **F** | **F** | **T** |

In instances of *modus tollens* we assume as premises that p → q is true and q is false. There is only one line of the truth table - the fourth line - which satisfies these two conditions. In this line, p is false. Therefore, in every instance in which p → q is true and q is false, p must also be false.